Scientific Mathematics Revision Notes

1 Key concepts

1.1 Simultaneous Linear Equations

We can solve 2 equations in 2 unknowns to find the values of the 2 unknowns.

1.1.1 Example

$$2x + 3y = 5.5 \quad (1)$$
$$x - 2y = 1 \quad (2)$$

Multiply equation (2) by 2.

$$2x - 4y = 2 \quad (3)$$

Now take (1)-(3) to find y

$$\begin{aligned} 2x + 3y &= 5.5 \\ 2x - 4y &= 2 \\ 0x + 7y &= 3.5 \\ y &= \frac{3.5}{7} \\ &= 0.5 \end{aligned}$$

Use this in (2) to find x

$$\begin{aligned} x - 2y &= 1 \\ x - 2 \times 0.5 &= 1 \\ x - 1 &= 1 \\ x &= 1 + 1 \\ x &= 2 \end{aligned}$$

Now put these values for x and y into the original equations to check they are a correct solution:

$$2x + 3y = 2 \times 2 + 3 \times 0.5 = 4 + 1.5 = 5.5 \quad \checkmark \quad (1)$$

$$x - 2y = 2 - 2 \times 0.5 = 2 - 1 = 1 \quad \checkmark \quad (2)$$

1.2 Quadratic Equations

A quadratic equation is one in x^2. The solution to a quadratic equation is finding the roots, values of x that satisfy the equation.

Some quadratic equations can be solved "by eye". But if you cannot spot a solution, quadratic equations can be solved using the quadratic formula.

If a quadratic equation is expressed

$$ax^2 + bx + c = 0$$

Then its roots are given by:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

1.2.1 Example

Solve the following quadratic equation

$$2x^2 - 6x + 4 = 0$$

In this case,

$$\begin{aligned} a &= 2 \\ b &= -6 \\ c &= 4 \end{aligned}$$

Using the quadratic formula,

$$\begin{aligned} x &= \frac{-(-6) \pm \sqrt{(-6)^2 - 4 \times 2 \times 4}}{2 \times 2} \\ &= \frac{6 \pm \sqrt{36 - 32}}{4} \\ &= \frac{6 \pm \sqrt{4}}{4} \\ &= \frac{6 \pm 2}{4} \\ &= \frac{6+2}{4} \quad or \quad \frac{6-2}{4} \\ &= 2 \quad or \quad 1 \end{aligned}$$

Check these values for x satisfy the original equation:

$$2x^2 - 6x + 4 = 0$$

Let $x = 1$:

$$2 \times 1^2 - 6 \times 1 + 4 = 2 - 6 + 4 = 0 \quad \checkmark$$

Let $x = 2$:

$$2 \times 2^2 - 6 \times 2 + 4 = 8 - 12 + 4 = 0 \quad \checkmark$$

So $x = 1$ and $x = 2$ are indeed roots of this quadratic.

2 General equation of a straight line

Every straight line can be expressed in the form

$$y = mx + c$$

c is the intercept (At $x = 0$, $y = c$).
m is the gradient (When x increases by a quantity a, y increases by $m \times a$. Or put another way, for every time x increases by 1, y increases by m).

Given two points on a straight line we can find values for m and c.

2.1 Example

On a straight line, at $x = 3$, $y = 0.5$ and at $x = 5$, $y = 1.5$. Find the equation of the line.

2.1.1 Method 1

As this is a straight line, it can be written in the form $y = mx + c$.

At $x = 3$,

$$m \times 3 + c = 0.5 \tag{1}$$

At $x = 5$,

$$m \times 5 + c = 1.5 \tag{2}$$

These can be solved as simultaneous equations.

2.1.2 Method 2

As this is a straight line, it can be written in the form $y = mx + c$. Since m is the gradient, we can calculate it as:

$$\begin{aligned} \text{gradient} = m &= \frac{\text{difference in } y \text{ values}}{\text{difference in } x \text{ values}} \\ &= \frac{1.5 - 0.5}{5 - 3} \\ &= \frac{1}{2} \end{aligned}$$

So now we have only one unknown and can solve this to find c, say at $(x, y) = (3, 0.5)$

$$\begin{aligned} y &= mx + c \\ 0.5 &= 0.5 \times 3 + c \\ \Longrightarrow c &= 0.5 - 0.5 \times 3 \\ &= -1 \end{aligned}$$

So $m = 0.5$, $c = -1$.

2.1.3 Method 3

Draw a graph, plot the two points $(x, y) = (3, 0.5)$ and $(x, y) = (5, 1.5)$ and draw a straight line between them. Read off the gradient and intercept (N.B. less accurate).

3 Indices

The following are the rules of indices:

$$x^m \times x^n = x^{m+n} \tag{1}$$

$$x^m \div x^n = \frac{x^m}{x^n} = x^{m-n} \tag{2}$$

$$(x^m)^n = x^{mn} \tag{3}$$

Note also that

$$\begin{aligned} x^0 &= 1 \\ x^{-n} &= \frac{1}{x^n} \\ x^{\frac{1}{q}} &= \sqrt[q]{x} \\ x^{\frac{p}{q}} &= \sqrt[q]{x^p} \end{aligned}$$

4 Exponential Function

$e \approx 2.718...$ is a special number used in the exponential function,

$$y = e^x$$

5 The Natural Logarithm Function

When $y = e^x$ is rearranged to give x in terms of y, the function which results is the natural logarithm function,

$$x = \ln y$$

$\ln a$ is $\log_e a$.

6 Logs to other bases

$\log_a b$ means logarithm to the base a of b. $c = log_a b$ is the number c which satisfies the equation $b = a^c$

6.1 Example

$$\begin{aligned} \log_{10} 100 = 2 &\quad \text{since } 10^2 = 100 \\ \log_2 256 = 8 &\quad \text{since } 2^8 = 256 \end{aligned}$$

7 Rules of Logarithms

$$\log_a A + \log_a B = \log_a AB \tag{1}$$

$$\log_a A - \log_a B = \log_a \frac{A}{B} \tag{2}$$

$$p \log_a A = \log_a A^p \tag{3}$$

8 Expontential Laws

Many natural processes can be modelled using a function such as,

$$y = Ae^{kx}$$

Such a function describes exponential growth $k > 0$ or decay $k < 0$. We can express such a function in linear form.

$$\begin{aligned}
y &= Ae^{kx} \\
\ln y &= \ln Ae^{kx} \\
\ln y &= \ln A + \ln e^{kx} \quad \text{(log rule 1)} \\
\ln y &= \ln A + kx \ln e \quad \text{(log rule 3)} \\
\ln y &= \ln A + kx \qquad \text{(since } \ln e = log_e e = 1\text{)} \\
\ln y &= \ln A + kx \\
\text{(cf.} \quad y &= c + mx \quad)
\end{aligned}$$

So k is the gradient and $\ln A$ the intercept of this straight line graph.

8.1 Example

We are told that a law is represented by

$$y = Ae^{kx}$$

And that at $t = 1$, $y = 0.2$ and at $t = 3$, $y = 0.7$.

Putting the law in linear form,

$$\ln y = \ln A + kt$$

At $t = 1$,

$$\ln 0.2 = \ln A + k \times 1 \tag{1}$$

At $t = 3$,

$$\ln 0.7 = \ln A + k \times 3 \tag{2}$$

Take (2) − (1)

$$\ln 0.2 = \ln A + k \tag{1}$$

$$\ln 0.7 = \ln A + 3k \tag{2}$$

$$(\ln 0.7 - \ln 0.2) = (\ln A - \ln A) + (3k - k) \tag{3}$$

$$\begin{aligned} \ln\frac{0.7}{0.2} &= 2k \\ \frac{\ln\frac{0.7}{0.2}}{2} &= k \ = 0.626(3 \text{ d.p.}) \end{aligned}$$

Using (1) we can find the value for A:

$$\begin{aligned} \ln 0.2 &= \ln A + k \\ \implies \ln A &= \ln 0.2 - k \\ &= -2.236(3 \text{ d.p.}) \\ \implies A &= e^{-2.236} = 0.107(3 \text{ d.p.}) \end{aligned}$$

9 Differentiation

Given a function $y = f(x)$, we can find the derivative $\frac{dy}{dx} = f'(x)$. This function enables us to find the gradient of the tangent to $y = f(x)$ at any point x, so describes the *rate of change* of $y = f(x)$.

9.1 Derivatives of some common functions

$y = f(x)$	$\frac{dy}{dx} = f'(x)$
ax	a
x^n	nx^{n-1}
$\ln x$	$\frac{1}{x}$
e^{kx}	ke^{kx}
$mf(x)$	$mf'(x)$
$f(x) + g(x)$	$f'(x) + g'(x)$

9.2 Examples

$$y = 3x^2 \qquad \frac{dy}{dx} = 6x$$

$$y = 2x^5 + 3x^2 + 1 \qquad \frac{dy}{dx} = 10x^4 + 6x$$

$$y = e^{3x} \qquad \frac{dy}{dx} = 3e^{3x}$$

$$y = 3\ln x \qquad \frac{dy}{dx} = \frac{3}{x}$$

$$y = \frac{1}{x^3} = x^{-3} \qquad \frac{dy}{dx} = -3x^{-4} = -\frac{3}{x^4}$$

$$y = \sqrt[3]{x} = x^{\frac{1}{3}} \qquad \frac{dy}{dx} = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3\sqrt[3]{x^2}}$$

$$y = \sqrt[4]{x^5} = x^{\frac{5}{4}} \qquad \frac{dy}{dx} = \frac{5}{4}x^{\frac{1}{4}} = \frac{1}{3\sqrt[4]{x}}$$

$$y = \frac{1}{\sqrt{x}} = x^{-\frac{1}{2}} \qquad \frac{dy}{dx} = -\frac{1}{2}x^{-\frac{3}{2}}$$

10 Integration

Integration can be thought of as the opposite process to differentiation, or as the process of finding the area under a curve by summation. For a function $f(x)$ we can find the indefinite integral,

$$F(x) = \int f(x)dx$$

Which is a function such that,

$$\frac{dF(x)}{dx} = f(x)$$

An indefinite integral always includes a constant of integration, since for c some constant, $\frac{d(c)}{dx} = 0$

10.1 Integrals of some common functions

$f(x)$	$\int f(x)dx$
a	$ax + c$
x^n	$\frac{1}{n+1}x^{n+1} + c$
$\frac{1}{x}$	$\ln x + c$
e^{kx}	$\frac{1}{k}e^{kx} + c$
$mf(x)$	$m\int f(x)dx$
$f(x) + g(x)$	$\int f(x)dx + \int g(x)dx$

10.1.1 Examples

$$\begin{aligned} y &= 2x^3 \\ \int ydx &= \tfrac{2}{4}x^4 + c \\ &= \tfrac{1}{2}x^4 + c \end{aligned}$$

$$\begin{aligned} y &= 3x^6 + 2x^4 + 1 \\ \int ydx &= \tfrac{3}{7}x^7 + \tfrac{2}{5}x^5 + x + c \end{aligned}$$

$$\begin{aligned} y &= 3e^{4x} \\ \int ydx &= \tfrac{3}{4}e^{4x} + c \end{aligned}$$

$$\begin{aligned} y &= \tfrac{4}{x} \\ \int ydx &= 4\ln x + c \end{aligned}$$

$$\begin{aligned} y &= \tfrac{1}{x^5} = x^{-5} \\ \int ydx &= -\tfrac{1}{4}x^{-4} + c \end{aligned}$$

$$\begin{aligned} y &= \sqrt{x} = x^{\frac{1}{2}} \\ \int ydx &= \tfrac{1}{\frac{3}{2}}x^{\frac{3}{2}} + c \\ &= \tfrac{2}{3}x^{\frac{3}{2}} + c \end{aligned}$$

$$\begin{aligned} y &= \sqrt[3]{x^2} = x^{\frac{2}{3}} \\ \int ydx &= \tfrac{1}{\frac{5}{3}}x^{\frac{5}{3}} + c \\ \int ydx &= \tfrac{3}{5}x^{\frac{5}{3}} + c \end{aligned}$$

$$\begin{aligned} y &= \tfrac{1}{\sqrt[3]{x}} = x^{-\frac{1}{3}} \\ \int ydx &= \tfrac{1}{\frac{2}{3}}x^{\frac{2}{3}} + c \\ \int ydx &= \tfrac{3}{2}x^{\frac{2}{3}} + c \end{aligned}$$

N.B. you can check whether you have correctly performed an integration by differentiating your answer and seeing if you get back to the question. Similarly, you can integrate a derivative to check it.

10.2 Definite Integration

This is the process of finding the value of an integral between two ordinates (values). If,

$$F(x) = \int f(x)dx$$

Then to find the definite integral between a and b we calculate

$$\begin{aligned}\int_b^a f(x)dx &= [F(x)]_a^b \\ &= F(b) - F(a)\end{aligned}$$

10.2.1 Example

Find the value of the definite integral,

$$\int_2^6 3x^6 + 2x^4 + 1dx$$

First find the integral,

$$\int_2^6 3x^6 + 2x^4 + 1dx = \left[\frac{3}{7}x^7 + \frac{2}{5}x^5 + x\right]_2^6$$

Then calculate the value of the integral at the limits and subtract,

$$\begin{aligned}\int_2^6 3x^6 + 2x^4 + 1dx &= \left[\frac{3}{7}x^7 + \frac{2}{5}x^5 + x\right]_2^6 \\ &= (\frac{3}{7} \times 6^7 + \frac{2}{5} \times 6^5 + 6) - (\frac{3}{7} \times 2^7 + \frac{2}{5} \times 2^5 + 2) \\ &= 123019.31 \quad \text{(2 d.p.)}\end{aligned}$$